LBL-39526
UCB-PTH-96/46

Finite dimensional unitary representations of quantum Anti–de Sitter groups at roots of unity

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Abstract

We study irreducible unitary representations of $U_q(SO(2,1))$ and $U_q(SO(2,3))$ for q a root of unity, which are finite dimensional. Among others, unitary representations corresponding to all classical one–particle representations with integral weights are found for $q = e^{i\pi/M}$, with M being large enough. In the "massless" case with spin bigger than or equal to 1 in 4 dimensions, they are unitarizable only after factoring out a subspace of "pure gauges", as classically. A truncated associative tensor product describing unitary many–particle representations is defined for $q = e^{i\pi/M}$.

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1 Introduction

In recent years, the development of Noncommutative Geometry has sparked much interest in formulating physics and in particular quantum field theory on quantized, i.e. noncommutative spacetime. The idea is that if there are no more "points" in spacetime, such a theory should be well behaved in the UV.

Quantum groups [8, 13, 7], although discovered in a different context, can be understood as generalized "symmetries" of certain quantum spaces. Thinking of elementary particles as irreducible unitary representations of the Poincaré group, it is natural to try to formulate a quantum field theory based on some quantum Poincaré group, i.e. on some quantized spacetime.

There have been many attempts (e.g. [25, 20]) in this direction. One of the difficulties with many versions of a quantum Poincaré group comes from the fact that the classical Poincaré group is not semisimple. This forbids using the well developed theory of (semi)simple quantum groups, which is e.g. reviewed in [2, 23, 11]. In this paper, we consider instead the quantum Anti–de Sitter group $U_q(SO(2,3))$, resp. $U_q(SO(2,1))$ in 2 dimensions, thus taking advantage of much well–known mathematical machinery. In the classical case, these groups (as opposed to e.g. the de Sitter group $SO(4,1)$) are known to have positive–energy representations for any spin [10], and e.g. allow supersymmetric extensions [29]. Furthermore, one could argue that the usual choice of flat spacetime is a singular choice, perhaps subject to some mathematical artefacts.

With this motivation, we study unitary representations of $U_q(SO(2,3))$. Classically, all unitary representations are infinite–dimensional since the group is noncompact. It is well known that at roots of unity, the irreducible representations (irreps) of quantum groups are finite dimensional. In this paper, we determine if they are unitarizable, and show in particular that for $q = e^{i\pi/M}$, all the irreps with positive energy and integral weights are unitarizable, as long as the rest energy $E_0 \geq s+1$ where s is the spin, and E_0 is below some (q – dependent, large) limit. There is an intrinsic high–energy cutoff, and only finitely many such "physical" representations exist for given q. At low energies and for q close enough to 1, the structure is the same as in the classical case. Furthermore, unitary representations exist only at roots of unity (if q is a phase). For generic roots of unity, their weights are non–integral. Analogous results are found for $U_q(SO(2,1))$. In general, there is a cell–like structure of unitary representations in weight space.

In the "massless" case, the naive representations with spin bigger than or equal to 1 are reducible and contain a null– subspace corresponding to "pure gauge" states. It is shown that they can be consistently factored out to obtain unitary representations with only the physical degrees of freedom ("helicities"), as in the classical case [10].

The existence of finite–dimensional unitary representations of noncompact quantum groups at roots of unity has already been pointed out in [6], where several representations of $U_q(SU(2,2))$ and $U_q(SO(2,3))$ (with multiplicity of weights equal to one) are shown to be unitarizable. In the latter case they correspond to the Dirac singletons [5], which are recovered here as well.

We also show that the class of "physical" (unitarizable) representations is closed under a new kind of associative truncated tensor product for $q = e^{i\pi/M}$, i.e. there exists a natural way to obtain many–particle representations.

Besides being very encouraging from the point of view of quantum field theory, this shows again the markedly different properties of quantum groups at roots of unity from the case of generic q and $q = 1$. The results are clearly not restricted to the groups considered here and should be of interest on purely mathematical grounds as well. We develop a method to investigate the structure of representations of quantum groups at roots of unity and determine the structure of a large class of representations of $U_q(SO(2,3))$. Throughout this paper, $U_q(SO(2,3))$ will be equipped with a non–standard Hopf algebra star structure.

The idea to find a quantum Poincaré group from $U_q(SO(2,3))$ is not new: Already in [20], the so–called κ–Poincaré group was constructed by a contraction of $U_q(SO(2,3))$. This contraction however essentially takes $q \to 1$ (in a nontrivial way) and destroys the properties of the representations which we emphasize, in particular the finite dimensionality.

Although it is not considered here, we want to mention that there exists a (space of functions on) quantum Anti–de Sitter space on which $U_q(SO(2,1))$ resp. $U_q(SO(2,3))$ operates, with an intrinsic mass parameter $m^2 = i(q - q^{-1})/R^2$ where R is the "radius" of Anti–de Sitter space (and the usual Minkowski signature for $q = 1$) [28].

This paper is organized as follows: In section 2, we investigate the unitary representations of $U_q(SO(2,1))$, and define a truncated tensor product. In section 3, the most important facts about quantized universal enveloping algebras of higher rank are reviewed. In section 4, we consider $U_q(SO(5))$ and $U_q(SO(2,3))$, determine the structure of the relevant irreducible representations (which are finite dimensional) and investigate which ones are unitarizable. The truncated tensor product is generalized to the case of $U_q(SO(2,3))$. Finally we conclude and look at possible further developments.

2 Unitary representations of $U_q(SO(2,1))$

We first consider the simplest case of $U_q(SO(2,1))$, which is a real form of $\mathcal{U} \equiv U_q(Sl(2,\mathbb{C}))$, the Hopf algebra defined by [8, 13]

$$[H, X^{\pm}] \;=\; \pm 2X^{\pm}, \quad [X^+, X^-] = [H] \tag{1}$$

$$
\begin{aligned}
\Delta(H) &= H \otimes 1 + 1 \otimes H, \\
\Delta(X^{\pm}) &= X^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X^{\pm}, \\
S(X^{+}) &= -qX^{+}, \quad S(X^{-}) = -q^{-1}X^{-}, \quad S(H) = -H \\
\varepsilon(X^{\pm}) &= \varepsilon(H) = 0
\end{aligned}
$$

where $[n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. To talk about a real form of $U_q(SL(2,\mathbb{C}))$, one has to impose a reality condition, i.e. a star structure, and there may be several possibilities. Since we want the algebra to be implemented by a unitary representation on a Hilbert space, the star operation should be an antilinear antihomomorphism of the algebra. Furthermore, we will see that to get finite dimensional unitary representations, q must be a root of unity, so $|q| = 1$. Only at roots of unity the representation theory of quantum groups differs essentially from the classical case, and new features such as finite dimensional unitary representations of noncompact groups can appear. This suggests the following star structure corresponding to $U_q(SO(2,1))$:

$$
H^* = H, \quad (X^+)^* = -X^- \tag{2}
$$

whis is simply

$$
x^* = e^{-i\pi H/2}\theta(x^{c.c.})e^{i\pi H/2} \tag{3}
$$

where θ is the usual (linear) Cartan–Weyl involution and $x^{c.c.}$ is the complex conjugate of $x \in \mathcal{U}$. Since q is a phase, $q^{c.c.} = q^{-1}$, and

$$
(\Delta(x))^* = \Delta(x^*) \tag{4}
$$

provided

$$
(a \otimes b)^* = b^* \otimes a^*. \tag{5}
$$

Then $(S(x))^* = S(x^*)$, which is a non–standard Hopf algebra star structure. In particular, (5) is chosen as e.g. in [24], which is different from the standard definition. Nevertheless, this is perfectly consistent with a many–particle interpretation in Quantum Mechanics or Quantum Field Theory as discussed in [28], where it is shown e.g. how to define an invariant inner product on the tensor product with the "correct" classical limit.

The irreps of $\mathcal{U}$ at roots of unity are well known (see e.g. [15], whose notations we largely follow), and we list some facts. Let

$$
q = e^{2\pi i n/m} \tag{6}
$$

for positive relatively prime integers m, n and define $M = m$ if m is odd, and $M = m/2$ if m is even. Then it is consistent and appropriate in our context to set

$$
(X^{\pm})^M = 0 \tag{7}
$$

(if one uses q^H instead of H, then $(X^{\pm})^M$ is central). All finite dimensional irreps are highest weight (h.w.) representations with dimension $d \leq M$. There are two types of irreps:

- $V_{d,z} = \{e^j_h; \quad j = (d-1) + \frac{m}{2n}z, \quad h = j, j-2, \ldots, -(d-1) + \frac{m}{2n}z\}$ with dimension d, for any $1 \leq d \leq M$ and $z \in \mathbb{Z}$, where $He^j_h = he^j_h$
- I^1_z with dimension M and h.w. $(M-1) + \frac{m}{2n}z$, for $z \in \mathbb{C} \setminus \{\mathbb{Z} + \frac{2n}{m}r, 1 \leq r \leq M-1\}$.

Note that in the second type, $z \in \mathbb{Z}$ is allowed, in which case we will write $V_{M,z} \equiv I^1_z$ for convenience. We will concentrate on the $V_{d,z}$ – representations from now on. Furthermore, the fusion rules at roots of unity state that $V_{d,z} \otimes V_{d',z'}$ decomposes into $\oplus_{d''} V_{d'',z+z'} \bigoplus_p I^p_{z+z'}$ where I^p_z are the well–known reducible, but indecomposable representations of dimension $2M$, see figure 1 and [15]. If q is *not* a root of unity, then the universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ given by

$$\mathcal{R} = q^{\frac{1}{2}H \otimes H} \sum_{l=0}^{\infty} q^{-\frac{1}{2}l(l+1)} \frac{(q-q^{-1})^l}{[l]!} q^{lH/2}(X^+)^l \otimes q^{-lH/2}(X^-)^l \tag{8}$$

defines the quasitriangular structure of $\mathcal{U}$. It satisfies e.g.

$$\sigma(\Delta(u)) = \mathcal{R}\Delta(u)\mathcal{R}^{-1}, \quad u \in \mathcal{U} \tag{9}$$

where $\sigma(a \otimes b) = b \otimes a$. We will only consider representations with dimension $\leq M$; then $\mathcal{R}$ restricted to such representations is well defined for roots of unity as well, since the sum in (8) only goes up to $(M-1)$. Furthermore

$$\mathcal{R}^* = (\mathcal{R})^{-1}. \tag{10}$$

To see this, (3) is useful.

Let us consider a hermitian invariant inner product (u, v) for $u, v \in V_{d,z}$. A hermitian inner product satisfies $(u, \lambda v) = \lambda(u, v) = (\lambda^{c.c.} u, v)$ for $\lambda \in \mathbb{C}$, $(u, v)^{c.c.} = (v, u)$, and it is invariant if

$$(u, x \cdot v) = (x^* \cdot u, v), \tag{11}$$

i.e. x^* is the adjoint of x. If $(\ ,\ )$ is also positive definite, we have a unitary representation.

Proposition 2.1 *The representations $V_{d,z}$ are unitarizable w.r.t. $U_q(SO(2,1))$ if and only if*

$$(-1)^{z+1} \sin(2\pi nk/m) \sin(2\pi n(d-k)/m) > 0 \tag{12}$$

for all $k = 1, ..., (d-1)$.

For $d-1 < \frac{m}{2n}$, this holds precisely if z is odd. For $d-1 \geq \frac{m}{2n}$, it holds for isolated values of d only, i.e. if it holds for d, then it (generally) does not *hold for $d \pm 1, d \pm 2, \ldots$.*

The representations $V_{d,z}$ are unitarizable w.r.t. $U_q(SU(2))$ if z is even and $d-1 < \frac{m}{2n}$.

Proof Let e^j_h be a basis of $V_{d,z}$ with highest weight j. After a straightforward calculation, invariance implies

$$\left((X^-)^k \cdot e^j_j, (X^-)^k \cdot e^j_j\right) = (-1)^k[k]![j][j-1]...[j-k+1]\left(e^j_j, e^j_j\right) \tag{13}$$

for $k = 1, ..., (d-1)$, where $[n]! = [1][2]...[n]$. Therefore we can have a positive definite inner product $(e^j_h, e^j_l) = \delta_{h,l}$ if and only if $a_k \equiv (-1)^k[k]![j][j-1]...[j-k+1]$ is a positive number for all $k = 1, ..., (d-1)$, in which case $e^j_{j-2k} = (a_k)^{-1/2}(X^-)^k \cdot e^j_j$.

Now $a_k = -[k][j-k+1]a_{k-1}$, and

$$\begin{aligned} -[k][j-k+1] &= -[k][d-k+\frac{m}{2n}z] = -[k][d-k]e^{i\pi z} && (14)\\ &= (-1)^{z+1}\sin(2\pi nk/m)\sin(2\pi n(d-k)/m)\frac{1}{sin(2\pi n/m)^2}, && (15) \end{aligned}$$

since z is an integer. Then the assertion follows. The compact case is known [15]. □

In particular, all of them are finite dimensional, and clearly if q is not a root of unity, none of the representations are unitarizable.

We will be particularly interested in the case of (half)integer representations of type $V_{d,z}$ and $n = 1, m$ even, for reasons to be discussed below. Then $d - 1 < \frac{m}{2n} = M$ always holds, and *the* $V_{d,z}$ *are unitarizable if and only if* z *is odd*. These representations are centered around Mz, with dimension less than or equal to M.

Let us compare this with the classical case. For the Anti–de Sitter group $SO(2,1)$, H is nothing but the energy (cp. section 3). At $q = 1$, the unitary irreps of $SO(2,1)$ are lowest weight representations with lowest weight $j > 0$ resp. highest weight representations with highest weight $j < 0$. For any given such lowest resp. highest weight we can now find a *finite dimensional* unitary representation with the same lowest resp. highest weight, provided M is large enough (we only consider (half)integer j here). These are unitary representations which for low energies look like the classical one– particle representations, but have an intrinsic high–energy cutoff if $q \neq 1$, which goes to infinity as $q \to 1$. The same will be true in the 4 dimensional case.

So far we only considered what could be called one–particle representations. To talk about many–particle representations, there should be a tensor product of 2 or more such irreps, which gives a unitary representation as well and agrees with the classical case for low energies.

Since $\mathcal{U}$ is a Hopf algebra, there is a natural notion of a tensor product of two representations, given by the coproduct Δ. However, it is not unitary a priori. As mentioned above,

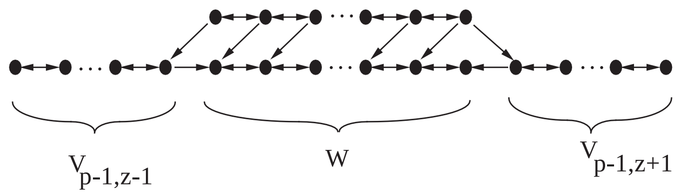

Figure 1: Indecomposable representation I_z^p

the tensor product of two irreps of type $V_{d,z}$ is

$$V_{d,z} \otimes V_{d',z'} = \oplus_{d''} V_{d'',z+z'} \bigoplus_{p=r,r+2,\ldots}^{d+d'-M} I^p_{z+z'} \tag{16}$$

where $r = 1$ if $d + d' - M$ is odd or else $r = 2$, and I_z^p is a indecomposable representation of dimenson $2M$ whose structure is shown in figure 1. The arrows indicate the rising and lowering operators.

In the case of $U_q(SU(2))$, one usually defines a truncated tensor product $\overline{\otimes}$ by omitting all indecomposable I_z^p representations [24]. Then the remaining representations are unitary w.r.t. $U_q(SU(2))$; $\overline{\otimes}$ is associative only from the representation theory point of view [24].

This is not the right thing to do for $U_q(SO(2,1))$. Let $n = 1$ and m even, and consider e.g. $V_{M-1,1} \otimes V_{M-1,1}$. Both factors have lowest energy $H = 2$, and the tensor product of the two corresponding *classical* representations is the sum of representations with lowest weights $4, 6, 8, \ldots$. In our case, these weights are in the I_z^p representations, while the $V_{d'',z''}$ have $H \geq M \to \infty$ and are not unitarizable. So we have to keep the I_z^p's and throw away the $V_{d'',z''}$'s in (16). A priori however, the I_z^p's are not unitarizable, either. To get a unitary tensor product, note that as a vector space,

$$I_z^p = V_{p-1,z-1} \oplus W \oplus V_{p-1,z+1} \tag{17}$$

(for $p \neq 1$) where

$$W = V_{M-p+1,z} \oplus V_{M-p+1,z} \tag{18}$$

as vector space. Now $(X^+)^{p-1} \cdot v_l$ is a lowest weight vector where v_l is the vector with lowest weight of I_p^z, and similarly $(X^-)^{p-1} \cdot v_h$ is a highest weight vector with v_h being the vector of I_p^z with highest weight (see figure 1). It is therefore consistent to consider the submodule of I_p^z generated by v_l, and factor out its submodule generated by $(X^+)^{p-1} \cdot v_l$; the result is an irreducible representation equivalent to $V_{p-1,z-1}$ realized on the left summand in (17).

Similarly, one could consider the submodule of I_p^z generated by v_h, factor out its submodule generated by $(X^-)^{p-1} \cdot v_h$, and obtain an irreducible representation equivalent to $V_{p-1,z+1}$. In short, one can just "omit" W in (17). The two V - type representations obtained this way are unitarizable provided $n = 1$ and m is even, and one can either keep both (notice the similarity with band structures in solid–state physics), or for simplicity keep the low–energy part only, in view of the physical application we have in mind. We therefore define a truncated tensor product as

Definition 2.2 *For $n = 1$ and even m,*

$$V_{d,z} \tilde{\otimes} V_{d',z'} := \bigoplus_{\tilde{d}=r,r+2,\ldots}^{d+d'-M} V_{\tilde{d},z+z'-1} \tag{19}$$

This can be stated as follows: Notice that any representation naturally decomposes as a vector space into sums of $V_{d,z}$'s, cp. (18); the definition of $\tilde{\otimes}$ simply means that only the smallest value of z in this decomposition is kept, which is the submodule of irreps with lowest weights less than or equal to $\frac{m}{2n}(z + z' - 1)$. (Incidentally, z is the eigenvalue of D_3 in the *classical* $su(2)$ algebra generated by $\{D^\pm = \frac{(X^\pm)^M}{[M]!}, 2D_3 = [D^+, D^-]\}$, where $\frac{(X^\pm)^M}{[M]!}$ is understood by some limes procedure). With this in mind, it is obvious that $\tilde{\otimes}$ is associative: both in $(V_1 \tilde{\otimes} V_2) \tilde{\otimes} V_3$ and in $V_1 \tilde{\otimes} (V_2 \tilde{\otimes} V_3)$, the result is simply the V's with minimal z, which is the *same* space, because the ordinary tensor product is associative and Δ is coassociative. This is in contrast with the "ordinary" truncated tensor product $\overline{\otimes}$ [24]. Of course, one could give a similar definition for negative–energy representations. See also definition 4.8 in the case of $U_q(SO(2,3))$.

$V_{d,z} \tilde{\otimes} V_{d',z'}$ is unitarizable if all the V 's on the rhs of (19) are unitarizable. This is certainly true if $n = 1$ and m is even. In all other cases, there are no terms on the rhs of (19) if the factors on the lhs are unitarizable, since no I_z^p – type representations are generated (they are too large). This is the reason why we concentrate on this case, and furthermore on $z = z' = 1$ which corresponds to low–energy representations. Then $\tilde{\otimes}$ defines a two–particle Hilbert space with the correct classical limit. To summarize, we have the following:

Proposition 2.3 *$\tilde{\otimes}$ is associative, and $V_{d,1} \tilde{\otimes} V_{d',1}$ is unitarizable.*

How an inner product is induced from the single – particle Hilbert spaces is explained in [28].

3 The quantum group $U_q(SO(2,3))$

In order to generalize the above results to the 4 – dimensional case, one has to use the general machinery of quantum groups, which is briefly reviewed (cp. e.g. [2]): Let $q \in \mathbb{C}$ and $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ be the Cartan matrix of a classical simple Lie algebra g of rank r, where $(,)$ is the Killing form and $\{\alpha_i, \quad i = 1,\ldots,r\}$ are the simple roots. Then the *quantized universal enveloping algebra* $U_q(g)$ is the Hopf algebra generated by the elements $\{X_i^{\pm}, H_i; \quad i = 1,\ldots,r\}$ and relations [8, 13, 7]

$$
\begin{aligned}
[H_i, H_j] &= 0 \\
\left[H_i, X_j^{\pm}\right] &= \pm A_{ji} X_j^{\pm}, \\
\left[X_i^{+}, X_j^{-}\right] &= \delta_{i,j}\frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{i,j}[H_i]_{q_i}, \\
\sum_{k=0}^{1-A_{ji}} & \left[\begin{array}{c} 1-A_{ji} \\ k \end{array}\right]_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-A_{ji}-k} = 0, \quad i \neq j
\end{aligned}
\tag{20}
$$

where $d_i = (\alpha_i,\alpha_i)/2, \quad q_i = q^{d_i}, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and

$$
\left[\begin{array}{c} n \\ m \end{array}\right]_{q_i} = \frac{[n]_{q_i}!}{[m]_{q_i}![n-m]_{q_i}!}. \tag{21}
$$

The comultiplication is given by

$$
\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\
\Delta(X_i^{\pm}) &= X_i^{\pm} \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_i^{\pm}.
\end{aligned}
\tag{22}
$$

Antipode and counit are

$$
\begin{aligned}
S(H_i) &= -H_i, \\
S(X_i^{+}) &= -q^{d_i} X_i^{+}, \quad S(X_i^{-}) = -q^{-d_i} X_i^{-}, \\
\varepsilon(H_i) &= \varepsilon(X_i^{\pm}) = 0.
\end{aligned}
\tag{23}
$$

(we use the conventions of [16], which differ slightly from e.g. [2].)

For $\mathcal{U} \equiv U_q(SO(5,\mathbb{C}))$, $r = 2$ and

$$
A_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \qquad (\alpha_i,\alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \tag{24}
$$

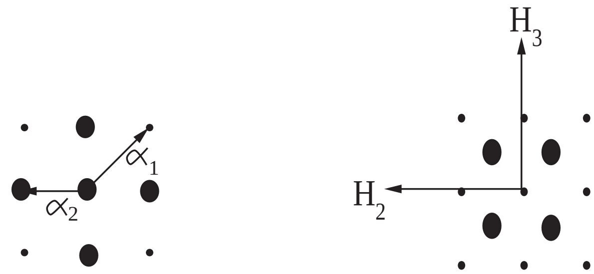

Figure 2: vector and spinor representations of $SO(2,3)$

so $d_1 = 1$, $d_2 = 1/2$, to have the standard physics normalization (a rescaling of (,) can be absorbed by a redefinition of q). The weight diagrams of the vector and the spinor representations are given in figure 2 for illustration. The Weyl element is $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \frac{3}{2}\alpha_1 + 2\alpha_2$.

The possible reality structures on $\mathcal{U}$ have been investigated in [19]. As in section 2, in order to obtain finite dimensional unitary representations, q must be a root of unity. Furthermore, on physical grounds we insist upon having positive–energy representations; already in the classical case, that rules out e.g. $SO(4,1)$, cp. the discussion in [10]. It appears that then there is only one possibility, namely

$$\begin{aligned} (H_i)^* &= H_i, \quad (X_1^+)^* = -X_1^-, \quad (X_2^+)^* = X_2^-, && (25)\\ (a \otimes b)^* &= b^* \otimes a^*, \\ (\Delta(u))^* &= \Delta(u^*), \quad (S(u))^* = S(u^*), && (26) \end{aligned}$$

for $|q| = 1$, which corresonds to the Anti–de Sitter group $U_q(SO(2,3))$. Again with $E \equiv d_1H_1 + d_2H_2$, $(-1)^E x^*(-1)^E = \theta(x^{c.c.})$ where θ is the usual Cartan–Weyl involution corresponding to $U_q(SO(5))$.

Although it will not be used it in the present paper, this algebra has the very important property of being *quasitriangular*, i.e. there exists a universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$. It satisfies $\mathcal{R}^* = (\mathcal{R})^{-1}$, which can be seen e.g. from uniqueness theorems, cp. [17, 2]. In the mathematical literature, usually a rational version of the above algebra, i.e. using $q^{d_iH_i}$ instead of H_i is considered. Since we are only interested in specific representations, we prefer to work with H_i. We essentially work in the "unrestricted" specialization, i.e. the divided powers $(X_i^{\pm})^{(k)} = \frac{(X_i^{\pm})^k}{[k]_{q_i}!}$ are not included if $[k]_{q_i} = 0$, although our results will only concern representations which are small enough so that the distinction is not relevant.

Often the following generators are more useful:

$$h_i = d_i H_i, \quad e_{\pm i} = \sqrt{[d_i]} X_i^{\pm}, \tag{27}$$

so that

$$\begin{aligned} [h_i, e_{\pm j}] &= \pm(\alpha_i, \alpha_j) e_{\pm j}, \\ [e_i, e_{-j}] &= \delta_{i,j} [h_i]. \end{aligned} \tag{28}$$

In the present case, i.e. $h_1 = H_1, h_2 = \frac{1}{2} H_2$, $e_{\pm 1} = X_1^{\pm}$ and $e_{\pm 2} = \sqrt{[\frac{1}{2}]} X_2^{\pm}$.

So far we only have the generators corresponding to the simple roots. A Cartan–Weyl basis corresponding to all roots can be obtained e.g. using the braid group action introduced by Lusztig [21], (see also [2, 11]) resp. the quantum Weyl group [16, 26, 18, 2]. If $\omega = \tau_{i_1} ... \tau_{i_N}$ is a reduced expression for the longest element of the Weyl group where τ_i is the reflection along α_i, then $\{\alpha_{i_1}, \tau_{i_1}\alpha_{i_2}, ..., \tau_{i_1}...\tau_{i_{N-1}}\alpha_{i_N}\}$ is an ordered set of positive roots. We will use $\omega = \tau_1\tau_2\tau_1\tau_2$ and denote them $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = \alpha_1 + \alpha_2, \beta_4 = \alpha_1 + 2\alpha_2$. A Cartan–Weyl basis of root vectors of $\mathcal{U}$ can then be defined as $\{e_{\pm 1}, e_{\pm 3}, e_{\pm 4}, e_{\pm 2}\} = \{e_{\pm 1}, T_1 e_{\pm 2}, T_1 T_2 e_{\pm 1}, T_1 T_2 T_1 e_{\pm 2}\}$ and similarly for the h_i 's, where the T_i represent the braid group on $\mathcal{U}$ [21]:

$$\begin{aligned} T_i(H_j) &= H_j - A_{ij} H_i, \quad T_i X_i^+ = -X_i^- q_i^{H_i}, \\ T_i(X_j^+) &= \sum_{r=0}^{-A_{ji}} (-1)^{r - A_{ji}} q_i^{-r} (X_i^+)^{(-A_{ji}-r)} X_j^+ (X_i^+)^{(r)} \end{aligned} \tag{29}$$

where $T_i(\theta(x^{c.c.})) = \theta(T_i(x))^{c.c.}$. We find

$$\begin{aligned} e_3 &= q^{-1} e_2 e_1 - e_1 e_2, \quad e_{-3} = q e_{-1} e_{-2} - e_{-2} e_{-1}, \quad h_3 = h_1 + h_2, \\ e_4 &= e_2 e_3 - e_3 e_2, \quad e_{-4} = e_{-3} e_{-2} - e_{-2} e_{-3}, \quad h_4 = h_1 + 2h_2. \end{aligned} \tag{30}$$

Similarly one defines the root vectors $X_{\beta_l}^{\pm}$. This can be used to obtain a Poincaré–Birkhoff–Witt basis of $\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+$ where $\mathcal{U}^{\pm}$ is generated by the $X_i^{\pm}$ and $\mathcal{U}^0$ by the H_i: for $\underline{k} := (k_1, \ldots, k_N)$ where N is the number of positive roots, let $X_{\underline{k}}^+ = X_{\beta_1}^{+k_1} \ldots X_{\beta_N}^{+k_N}$. Then the $X_{\underline{k}}^{\pm}$ form a P.B.W. basis of $\mathcal{U}^+$, and similarly for $\mathcal{U}^-$ [22] (assuming $q^4 \neq 1$).

Up to a trivial automorphism, (30) agrees with the basis used in [20]. The identification of the usual generators of the Poincaré group has also been given there and will not be repeated here, except for pointing out that h_3 is the energy and h_2 is a component of angular

momentum, see also [10]. All of the above form $U_{\tilde{q}}(SL(2,\mathbb{C}))$ subalgebras with appropriate $\tilde{q}$ (but not as coalgebras), because the T_i's are algebra homomorphisms. The reality structure is

$$e_1^* = -e_{-1}, \quad e_2^* = e_{-2}, \quad e_3^* = -e_{-3}, \quad e_4^* = -e_{-4}. \tag{31}$$

So the set $\{e_{\pm 2}, h_2\}$ generates a $U_{\tilde{q}}(SU(2))$ algebra, and the other three $\{e_{\pm\alpha}, h_\alpha\}$ generate noncompact $U_{\tilde{q}}(SO(2,1))$ algebras, as discussed in section 2.

4 Unitary representations of $U_q(SO(2,3))$ and $U_q(SO(5))$

In this section, we consider representations of $U_q(SO(2,3))$ and show that for suitable roots of unity q, the irreducible positive resp. negative energy representations are again unitarizable, if the highest resp. lowest weight lies in some "bands" in weight space. Their structure for low energies is exactly as in the classical case including the appearance of "pure gauge" subspaces for spin bigger than or equal to 1 in the "massless" case, which have to be factored out to obtain the physical, unitary representations. At high energies, there is an intrinsic cutoff.

From now on $q = e^{2\pi i n/m}$. Most facts about representations of quantum groups we will use can be found e.g. in [4]. It is useful to consider the Verma modules $M(\lambda)$ for a highest weight λ, which is the (unique) $\mathcal{U}$ - module having a highest weight vector w_λ such that

$$\mathcal{U}^+ w_\lambda = 0, \quad H_i w_\lambda = \frac{(\lambda, \alpha_i)}{d_i} w_\lambda, \tag{32}$$

and the vectors $X^-_{\underline{k}} w_\lambda$ form a P.B.W. basis of $M(\lambda)$. On a Verma module, one can define a unique invariant inner product (,), which is hermitian and satisfies $(w_\lambda, w_\lambda) = 1$ and $(u, x \cdot v) = (\theta(x^{c.c.}) \cdot u, v)$ for $x \in \mathcal{U}$, as in section 2 [4]. θ is again the (linear) Cartan – Weyl involution corresponding to $U_q(SO(5))$.

The irreducible highest weight representations can be obtained from the corresponding Verma module by factoring out all submodules in the Verma module. All submodules are null spaces w.r.t. the above inner product, i.e. they are orthogonal to any state in $M(\lambda)$. Therefore one can consistently factor them out, and obtain a hermitian inner product on the quotient space $L(\lambda)$, which is the unique irrep with highest weight λ. To see that they are null, let $w_\mu \in M(\lambda)$ be in some submodule, so $w_\lambda \notin \mathcal{U}^+ w_\mu$. Now for $v \in \mathcal{U}^- w_\lambda$, it follows $(w_\mu, v) \in (\mathcal{U}^+ w_\mu, w_\lambda) = 0$.

The following discussion until the paragraph before Definition 4.4 is technical and may be skipped upon first reading. Let $Q = \sum \mathbb{Z}\alpha_i$ be the root lattice and $Q^+ = \sum \mathbb{Z}_+\alpha_i$ where

$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We will write

$$\lambda \succ \mu \quad \text{if} \quad \lambda - \mu \in Q^+. \tag{33}$$

For $\eta \in Q$, denote (see [4])

$$\mathrm{Par}(\eta) := \{\underline{k} \in \mathbb{Z}_+^N; \quad \sum k_i \beta_i = \eta\}. \tag{34}$$

Let $M(\lambda)_\eta$ be the weight space with weight $\lambda - \eta$ in $M(\lambda)$. Then its dimension is given by $|\mathrm{Par}(\eta)|$. If $M(\lambda)$ contains a highest weight vector with weight σ, then the multiplicity of the weight space $(M(\lambda)/M(\sigma))_\eta$ is given by $|\mathrm{Par}(\eta)| - |\mathrm{Par}(\eta + \sigma - \lambda)|$, and so on. We will see how this allows to determine the structure, i.e. the characters of the irreducible highest weight representations.

As usual, the character of a representation $V(\lambda)$ with maximal weight λ is the function on weight space defined by

$$\mathrm{ch}(V(\lambda)) = e^\lambda \sum_{\eta \in Q^+} \dim V(\lambda)_\eta e^{-\eta}, \tag{35}$$

where $e^{\lambda-\eta}(\mu) := e^{(\lambda-\eta,\mu)}$, and $V(\lambda)_\eta$ is the weight space of $V(\lambda)$ at weight $\lambda - \eta$. The characters of inequivalent highest weight irreps (which are finite dimensional at roots of unity) are linearly independent. Furthermore, the characters of Verma modules are the same as in the classical case [12, 4],

$$\mathrm{ch}(M(\lambda)) = e^\lambda \sum_{\eta \in Q^+} |\mathrm{Par}(\eta)| e^{-\eta}. \tag{36}$$

In general, the structure of Verma modules is quite complicated, and the proper technical tool to describe it is its *composition series*. For a $\mathcal{U}$ –module M with a maximal weight, consider a sequence of submodules $\ldots \subset W_2 \subset W_1 \subset W_0 = M$ such that W_k/W_{k+1} is irreducible, and thus $W_k/W_{k+1} \cong L(\mu_k)$ for some μ_k. (If the series is finite, it is sometimes called a Jordan–Hölder series. For roots of unity it is infinite, but this is not a problem for our arguments. W_{k+1} can be constructed inductively by fixing a maximal submodule of W_k, e.g. as the sum of all but one highest weight submodules of W_k). While the submodules W_k may not be unique, it is obvious that we always have $\mathrm{ch}(M) = \sum \mathrm{ch}(W_k/W_{k+1}) = \sum \mathrm{ch}(L(\mu_k))$. Since the characters of irreps are linearly independent, this decomposition of $\mathrm{ch}(M)$ is unique, and so are the subquotients $L(\mu_k)$. We will study the composition series of the Verma module $M(\lambda)$, in order to determine the structure of the corresponding irreducible highest weight representation.

Our main tool to achieve this is a remarkable formula by De Concini and Kac for $\det(M(\lambda)_\eta)$, the determinant of the inner product matrix of $M(\lambda)_\eta$. Before stating it, we point out its use for determining irreps:

Lemma 4.1 *Let w_λ be the highest weight vector in an irreducible highest weight representation $L(\lambda)$ with invariant inner product. If $(w_\lambda, w_\lambda) \neq 0$, then $(\ ,\ )$ is non–degenerate, i.e.*

$$det(L(\lambda)_\eta) \neq 0 \tag{37}$$

for every weight space with weight $\lambda - \eta$ in $L(\lambda)$.

Proof Assume to the contrary that there is a vector v_μ which is orthogonal to all vectors of the same weight, and therefore to all vectors of any weight. Because $L(\lambda)$ is irreducible, there exists an $u \in \mathcal{U}$ such that $w_\lambda = u \cdot v_\mu$. But then $(w_\lambda, w_\lambda) = (w_\lambda, u \cdot v_\mu) = (u^\dagger \cdot w_\lambda, v_\mu) = 0$, which is a contradiction. $\square$

Now we state the result of De Concini and Kac [4]:

$$\det(M(\lambda)_\eta) = \prod_{\beta \in R^+} \prod_{m_\beta \in I\!N} \left([m_\beta]_{d_\beta} \frac{q^{(\lambda+\rho-m_\beta\beta/2,\beta)} - q^{-(\lambda+\rho-m_\beta\beta/2,\beta)}}{q^{d_\beta} - q^{-d_\beta}} \right)^{|\mathrm{Par}(\eta - m_\beta\beta)|} \tag{38}$$

in a P.B.W. basis for arbitrary highest weight λ, where R^+ denotes the positive roots (cp. section 3), and $d_\beta = (\beta, \beta)/2$.

To get some insight, notice first of all that due to $|\mathrm{Par}(\eta - m_\beta\beta)|$ in the exponent, the product is finite. Now for some positive root β, let k_β be the smallest integer such that $D(\lambda)_{k_\beta,\beta} := \left([k_\beta]_{d_\beta} \frac{q^{(\lambda+\rho-k_\beta\beta/2,\beta)} - q^{-(\lambda+\rho-k_\beta\beta/2,\beta)}}{q^{d_\beta} - q^{-d_\beta}} \right) = 0$, and consider the weight space at weight $\lambda - k_\beta\beta$, i.e. $\eta_\beta = k_\beta\beta$. Then $|\mathrm{Par}(\eta_\beta - k_\beta\beta)| = 1$ and $\det(M(\lambda)_{\eta_\beta})$ is zero, so there is a highest weight vector w_β with weight $\lambda - \eta_\beta$ (assuming for now that there is no other null state with weight larger than $(\lambda - \eta_\beta)$). It generates a submodule which is again a Verma module (because $\mathcal{U}$ does not have zero divisors [4]), with dimension $|\mathrm{Par}(\eta - k_\beta\beta)|$ at weight $\lambda - \eta$. This is the origin of the exponent. However the submodules generated by the ω_{β_i} are in general not independent, i.e. they may contain common highest weight vectors, and other highest weight vectors besides these w_{β_i} might exist. Nevertheless, all the highest weights μ_k in the composition series of $M(\lambda)$ are precisely obtained in this way. This "strong linkage principle" will be proven below, adapting the arguments in [12] for the classical case. While it is not a new insight for the quantum case either [6, 1], it seems that no explicit proof has been given at least in the case of even roots of unity, which is most interesting from our point of view, as we will see.

To make the structure more transparent, let $I\!N_\beta^T$ be the set of positive integers k with $[k]_\beta = 0$, and $I\!N_\beta^R$ the positive integers k such that $(\lambda+\rho-\frac{k}{2}\beta,\beta) \in \frac{m}{2n}Z\!\!Z$. Then

$$D(\lambda)_{k,\beta} = 0 \Leftrightarrow k \in I\!N_\beta^T \quad \text{or} \quad k \in I\!N_\beta^R. \tag{39}$$

The second condition is $k = 2\frac{(\lambda+\rho,\beta)}{(\beta,\beta)} + \frac{m}{2n}\frac{2}{(\beta,\beta)}Z\!\!Z$, which means that

$$\lambda - k\beta = \sigma_{\beta,l}(\lambda) \tag{40}$$

where $\sigma_{\beta,l}(\lambda)$ is the reflection of λ by a plane perpendicular to β through $-\rho + \frac{m}{4nd_\beta}l\beta$, for some integer l. For general l, $\sigma_{\beta,l}(\lambda) \notin \lambda + Q$; but k should be an integer, so it is natural to define the *affine Weyl group* $\mathcal{W}_\lambda$ of reflections in weight space to be generated by those reflections σ_{β_i,l_i} in weight space which map λ into $\lambda + Q$. For $q = e^{2\pi i n/m}$, two such allowed reflection planes perpendicular to β_i will differ by multiples of $\frac{1}{2}M_{(i)}\beta_i$; here $M_{(i)} = m$ for $d_i = \frac{1}{2}$, while for $d_i = 1$, $M_{(i)} = m$ or $m/2$ if m is odd or even, respectively. Thus $\mathcal{W}_\lambda$ is generated by all reflections by these planes. Alternatively, it is generated by the usual Weyl group with reflection center $-\rho$, and translations by $M_{(i)}\beta_i$.

Now the *strong linkage principle* states the following:

Proposition 4.2 *$L(\mu)$ is a composition factor of the Verma module $M(\lambda)$ if and only if μ is* strongly linked *to λ, i.e. if there is a descendant sequence of weights related by the affine Weyl group as*

$$\lambda \succ \lambda_i = \sigma_{\beta_i,l_i}(\lambda) \succ \ldots \succ \lambda_{kj\ldots i} = \sigma_{\beta_k,l_k}(\lambda_{j\ldots i}) = \mu \tag{41}$$

Proof The main tool to show this is the formula (38). Consider the inner product matrix $M_{\underline{k},\underline{k}'} := (X^-_{\underline{k}}w_\lambda, X^-_{\underline{k}'}w_\lambda)$; it is hermitian, since q is a phase. One can define an analytic continuation of it as follows: for the same P.B.W. basis, let $B_{\underline{k},\underline{k}'}(q,\lambda) := (X^-_{\underline{k}}w_\lambda, X^-_{\underline{k}'}w_\lambda)_b$ be the matrix of the invariant *bilinear* form defined as in [4], which is manifestly analytic in q and λ (one considers q as a formal variable and replaces $q \rightarrow q^{-1}$ in the first argument of $(\ ,\ )_b$). Then (38) holds for all $q \in C$ and arbitrary complexified λ [4]. For $|q| = 1$ and real λ, $B_{\underline{k},\underline{k}'}(q,\lambda) = M_{\underline{k},\underline{k}'}$. Let $\lambda' = \lambda + h\rho$ and $q' = qe^{i\pi h}$ for $h \in C$; then $B_{\underline{k},\underline{k}'}(q',\lambda')$ is analytic in h, and hermitian for $h \in I\!R$. Furthermore, one can identify the modules $M(\lambda')$ for different h via the P.B.W. basis. In this sense, the action of $X_i^\pm$ is analytic in h (it only depends on the commutation relations of the $X_\beta^\pm$). Now it follows (see theorem 1.10 in [14], chapter 2 on matrices which are analytic in h and normal for real h) that the eigenvalues e_j of $B_{\underline{k},\underline{k}'}(q',\lambda')$ are analytic in h, and there exist analytic projectors P_{e_j} on the eigenspaces V_{e_j}

which span the entire vectorspace (except possibly at isolated points where some eigenvalues coincide; for $h \in I\!R$ however, the generic eigenspaces are orthogonal and therefore remain independent even at such points). These projectors provide an analytic basis of eigenvectors of $B_{\underline{k},\underline{k}'}(q', \lambda')$. Now let

$$V_k := \bigoplus_{e_j \propto h^k} V_{e_j}, \tag{42}$$

i.e. the sum of the eigenspaces whose eigenvalues e_j have a zero of order k (precisely) at $h = 0$. Of course, $(V_k, V_{k'})_b = 0$ for $k \neq k'$. The V_k span the entire space, they have an analytic basis as discussed, and have the following properties:

Lemma 4.3 *1) $(v_k, v)_b = o(h^k)$ for $v_k \in V_k$ and* any *(analytic) $v \in M(\lambda')$.*

2) $X_i^{\pm} v_k = \sum_{l \geq k} a_l v_l + \sum_{l=1}^{k} h^l b_l v_{k-l}$ for $v_l \in V_l$ and a_l, b_l analytic. In particular at $h = 0$,

$$M^k := \oplus_{n \geq k} V_n \tag{43}$$

is invariant.

Proof

1) Decomposing v according to $\oplus_l V_l$, only the (analytic) component in V_k contributes in $(v_k, v)_b$, with a factor h^k by the definition of V_k ($o(h^k)$ means at least k factors of h).

2) Decompose $X_i^{\pm} v_k = \sum_{e_j} a_{e_j} v_{e_j}$ with analytic coefficients a_{e_j} corresponding to the eigenvalue e_j. For any v_{e_j} appearing on the rhs, consider $(v_{e_j}, X_i^{\pm} v_k)_b = a_{e_j}(v_{e_j}, v_{e_j})_b = c\ a_{e_j} e_j$ (with $c \neq 0$ at $h = 0$, since v_{e_j} might not be normalized). But the lhs is $(X_i^{\mp} v_{e_j}, v_k)_b = o(h^k)$ as shown above. Therefore $a_{e_j} e_j = o(h^k)$, which implies 2).

□

In particular, $M(\lambda)/_{M^1}$ is irreducible and nothing but $L(\lambda)$. (The sequence of submodules $... \subset M^2 \subset M^1 \subset M(\lambda)$ is similar to the Jantzen filtration [12].)

By the definition of M^k, we have

$$\mathrm{ord}(\det(M(\lambda)_\eta)) = \sum_{k \geq 1} \dim M^k_{\lambda - \eta} \tag{44}$$

where $M^k_{\lambda-\eta}$ is the weight space of M^k at weight $\lambda - \eta$, and $\mathrm{ord}(\det(M(\lambda)_\eta))$ is the order of the zero of $\det(M(\lambda)_\eta)$ as a function of h, i.e. the maximal power of h it contains. Now from (38), it follows that

$$
\begin{aligned}
\sum_{k\geq 1} \mathrm{ch}(M^k) &= e^\lambda \sum_{\eta\in Q^+} (\sum_{k\geq 1} \dim M^k_{\lambda-\eta}) e^{-\eta} \\
&= e^\lambda \sum_{\eta\in Q^+} \mathrm{ord}(\det(M(\lambda)_\eta)) e^{-\eta} \\
&= \sum_{\beta\in I\!R^+} (\sum_{n\in I\!N^T_\beta} + \sum_{n\in I\!N^R_\beta}) e^\lambda \sum_{\eta\in Q^+} |\mathrm{Par}(\eta - n\beta)| e^{-\eta} \\
&= \sum_{\beta\in I\!R^+} (\sum_{n\in I\!N^T_\beta} + \sum_{n\in I\!N^R_\beta}) \mathrm{ch}(M(\lambda - n\beta))
\end{aligned}
\tag{45}
$$

where we used (36).

Now we can show (4.2) inductively. Both the left and the right side of (45) can be decomposed into a sum of characters of highest weight irreps, according to their composition series. These characters are linearly independent. Suppose that $L(\lambda - \eta)$ is a composition factor of $M(\lambda)$. Then the corresponding character is contained in the lhs of (45), since $M(\lambda)/_{M^1}$ is irreducible. Therefore it is also contained in one of the $\mathrm{ch}(M(\lambda - n\beta))$ on the rhs. Therefore $L(\lambda - \eta)$ is a composition factor of one of these $M(\lambda - n\beta)$, and by the induction assumption we obtain that $\mu \equiv \lambda - \eta$ is strongly linked to λ as in (41).

Conversely, assume that μ satisfies (41). By the induction assumption, there exists a $n \in I\!N^T_\beta \cup I\!N^R_\beta$ such that $L(\mu)$ is a subquotient of $M(\lambda - n\beta)$. Then (45) shows that $L(\mu)$ is a subquotient of $M(\lambda)$. $\square$

Obviously this applies to other quantum groups as well. In particular, we recover the well–known fact that for $q = e^{2\pi i n/m}$, all $(X_i^-)^{M_{(i)}} w_\lambda$ are highest weight vectors, and zero in an irrep.

Now we can study the irreps of $U_q(SO(5))$ and $U_q(SO(2,3))$. First, there exist remarkable nontrivial one dimensional representations w_{λ_0} with weights $\lambda_0 = \sum \frac{m}{2n} k_i \alpha_i$ for integers k_i. By tensoring any representation with w_{λ_0}, one obtains another representation with the same structure, but all weights shifted by λ_0. We will see below that by such a shift, representations which are unitarizable w.r.t. $U_q(SO(2,3))$ are in one to one correspondence with representations which are unitarizable w.r.t. $U_q(SO(5))$. It is therefore enough to consider highest weights in the following domain:

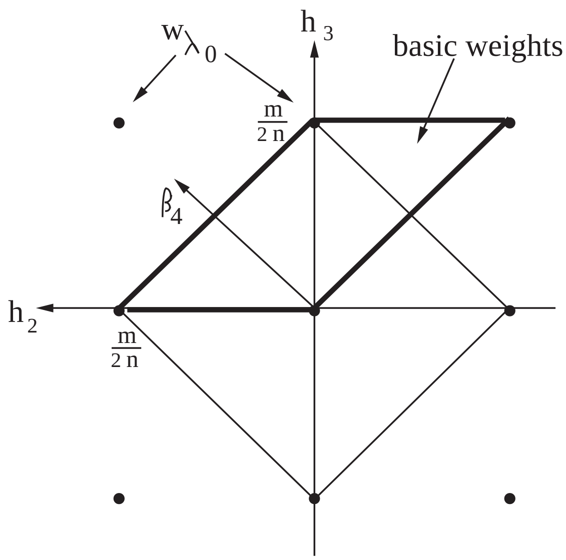

Figure 3: envelope of compact representations, basic weights and the lattice of w_{λ_0}

Definition 4.4 *A weight* $\lambda = E_0\beta_3 + s\beta_2$ *is called* basic *if*

$$0 \leq (\lambda, \beta_3) = E_0 < \frac{m}{2n}, \quad 0 \leq (\lambda, \beta_4) = (E_0 + s) < \frac{m}{2n}. \tag{46}$$

In particular, $\lambda \succ 0$*. It is said to be* compact *if in addition it is integral (i.e.* $(\lambda, \beta_i) \in \mathbb{Z} d_i$*),*

$$s \geq 0 \quad \textit{and} \quad (\lambda, \beta_1) \geq 0. \tag{47}$$

An irrep with compact highest weight will be called compact.

The region of basic weights is drawn in figure 3, together with the lattice of w_{λ_0}'s. The compact representations are centered around 0, and the (quantum) Weyl group acts on them [16], as classically (it is easy to see that the action of the quantum Weyl group resp. braid group on the compact representations is well defined at roots of unity as well).

A representation with basic highest weight can be unitarizable w.r.t. $U_q(SO(5))$ (with conjugation $\theta^{c.c.}$) only if all the $U_q(SU(2))$'s are unitarizable. For compact λ, all the $U_q(SU(2))$'s are indeed unitarizable according to section 2, where $M_{(2)} = M_{(3)} = m$ and $M_{(1)} = M_{(4)} = m$ or $m/2$ if m is odd or even, respectively. This alone however is not enough to show that they are unitarizable w.r.t. to the full group.

Although it may not be expected, there exist unitary representations with non–integral basic highest weight, namely for

$$\lambda = \frac{m-1}{2}\beta_3 \quad \text{and} \quad \lambda = (\frac{m}{2} - 1)\beta_3 + \frac{1}{2}\beta_2, \tag{48}$$

if $n = 1$ and m even. It follows from Proposition 4.2 that they contain a highest weight vector at $\lambda - 2\beta_3$ and $\lambda - \beta_3$ respectively, and all the multiplicities in the irreps turn out to be one. Furthermore all $U_q(SU(2))$ modules in β_1, β_4 direction have maximal length $M_{(1)} = m/2$, which implies that they are unitarizable. The structure is that of shifted Dirac singletons which were already studied in [6], and we will return to this later.

It appears that all other irreps must have integral highest weight in order to be unitarizable w.r.t. $U_q(SO(5))$. If the highest weight is not compact, some of the $U_q(SU(2))$'s will not be unitarizable. On the other hand, all irreps with compact highest weight are indeed unitarizable:

Theorem 4.5 *The structure of the irreps $V(\lambda)$ with compact highest weight λ is the same as in the classical case except if*

a) $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ for $s \geq 1$ and $\frac{m}{2n}$ integer, where one additional highest weight state at weight $\lambda - \beta_4$ appears and no others, and

b) $\lambda = \frac{m-1}{2}\beta_3$ and $\lambda = (\frac{m}{2} - 1)\beta_3 + \frac{1}{2}\beta_2$ for $n = 1$ and m odd, where one additional highest weight state at weight $\lambda - 2\beta_3$ resp. $\lambda - \beta_3$ appears and no others,

which are factored out in the irrep. They are unitarizable w.r.t. $U_q(SO(5))$ (with conjugation $\theta^{c.c.}$).

The irreps with nonintegral highest weights (48) discussed above are unitarizable as well.

Proof The statements on the structure follow easily from Proposition 4.2.

To show that these irreps are unitarizable, consider the compact representation with highest weight λ before factoring out the additional highest weight state, so that the space is the same as classically. For $q = 1$, they are known to be unitarizable, so the inner product is positive definite. Consider the eigenvalues of the (hermitian) inner product matrix $M_{\underline{k},\underline{k}'}$ as q goes from 1 to $e^{2\pi i n/m}$ along the unit circle. The only way an eigenvalue could become negative is that it is zero for some $q' \neq q$. This can only happen if q' is a root of unity, $q' = e^{2i\pi n'/m'}$ with $n'/m' < n/m$. then the "non–classical" reflection planes of $\mathcal{W}_\lambda$ are further away from the origin and are relevant only in the case $\lambda = \frac{m-1}{2}\beta_3$ for $n = 1$ and m odd; but as pointed out above, no additional eigenvector appears in this case for $q' \neq q$.

Thus the eigenvalues might only become zero at q. This happens precisely if a new highest weight vector appears, i.e. in the cases listed. Since there is no null vector in the remaining irrep, all its eigenvalues are positive by continuity. $\square$

So far, all results were stated for highest weight modules; of course the analogous statements for lowest weight modules are true as well.

Now we want to find the "physical", positive–energy representations which are unitarizable w.r.t. $U_q(SO(2,3))$. They are most naturally considered as lowest weight representations, and can be obtained from the compact case by a shift, as indicated above: if $V(\lambda)$ is a compact highest weight representation, then

$$V(\lambda) \cdot \omega \ := V(\lambda) \otimes \omega \tag{49}$$

with $\omega \equiv w_{\lambda_0}, \lambda_0 = \frac{m}{2n}\beta_3$ has lowest weight $\mu = -\lambda + \lambda_0 \equiv E_0\beta_3 - s\beta_2$ (short: $\mu = (E_0, s)$). It is a positive–energy representation, i.e. the eigenvalues of h_3 are positive.

For $\frac{m}{2n}$ integer, these representations will correspond precisely to classical positive–energy representations with the same lowest weight [10]. The states with smallest energy h_3 corresponds to the particle at rest, so E_0 is the rest energy and s the spin. For $h_3 \leq m/4n$, the structure is the same as classically, see figure 4. The irreps with nonintegral highest weights (48) upon this shift correspond to the Dirac singleton representations "Rac" with lowest weight $\mu = (1/2, 0)$ and "Di" with $\mu = (1, 1/2)$, as discussed in [6].

If $\frac{m}{2n}$ is not integer, the weights of shifted compact representations are not integral. For $n = 1$ and m odd, the irreps in b) of theorem (4.5) now correspond to the singletons, again in argeement with [6]. We will see however that this case does not lead to an interesting tensor product.

The case $\mu = (s+1, s)$ for $s \geq 1$ and $\frac{m}{2n}$ integer will be called "massless" for two reasons. First, E_0 is the smallest possible rest energy for a unitarizable representation with given s (see below). The main reason however is the fact that as in the classical case [10], an additional lowest weight state with $E_0' = E_0 + 1$ and $s' = s - 1$ appears, which generates a null subspace of what should be called "pure gauge" states. This corresponds precisely to the classical phenomenon in gauge theories, which ensures that the massless photon, graviton etc. have only their appropriate number of degrees of freedom (generally, the concept of mass in Anti–de Sitter space is not as clear as in flat space. Also notice that while "at rest" there are actually still $2s + 1$ states, the representation is nevertheless reduced by one irrep of spin $s - 1$). In the present case, all these representations are finite–dimensional!

Thus we are led to the following.

Definition 4.6 *An irreducible representation $V_{(\mu)}$ with lowest weight $\mu = (E_0, s) \equiv E_0\beta_3 - s\beta_2$ (resp. μ itself) is called* physical *if it is unitarizable w.r.t. $U_q(SO(2,3))$ (with conjugation as in (26)).*

It is called massless *if $E_0 = s + 1$ for $s \geq 1$, $s \in \frac{1}{2}\mathbb{Z}$ and $\frac{m}{2n}$ integer.*

For $n = 1$, $V_{(\mu)}$ is called Di *if $\mu = (1, 1/2)$ and* Rac *if $\mu = (1/2, 0)$.*

Theorem 4.7 *The irreducible representation $V_{(\mu)}$ with lowest weight μ is physical, i.e. unitarizable w.r.t. $U_q(SO(2,3))$, if and only if the (shifted) irreducible represesentation with lowest weight $\mu - \frac{m}{2n}\beta_3$ is unitarizable w.r.t. $U_q(SO(5))$.*

All $V_{(\mu)}$ where $-(\mu - \frac{m}{2n}\beta_3)$ is compact are physical, as well as the singletons Di and Rac. For $h_3 \leq \frac{m}{4n}$, $V_{(\mu)}$ is obtained by factoring out from a (lowest weight) Verma module a submodule with lowest weight $(E_0, -(s+1))$, except for the massless case, where one additional lowest weight state with weight $(E_0+1, s-1)$ appears, and for the Di and Rac, where one additional lowest weight state with weight (E_0+1, s) and (E_0+2, s) appears, respectively. This is the same as for $q = 1$, see figure 4.

For the singletons, this was already shown in [6].

Proof As mentioned before, we can write every vector in such a representation uniquely as $a \cdot \omega$, where a belongs to a unitarizable irrep of $U_q(SO(5))$. Consider the inner product

$$\langle a \cdot \omega, b \cdot \omega \rangle \equiv (a, b), \tag{50}$$

where (a, b) is the hermitian inner product on the *compact* (shifted) representation. Then

$$\begin{aligned} \langle a \cdot \omega, e_1(b \cdot \omega) \rangle &= \langle a \cdot \omega, (e_1 \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_1) b \otimes \omega \rangle \\ &= q^{h_1/2}|_\omega (a, e_1 b) = i(a, e_1 b) \end{aligned} \tag{51}$$

using $h_1|_\omega = \frac{m}{2n}$. Similarly,

$$\begin{aligned} \langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle &= \langle (e_{-1} \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_{-1}) a \otimes \omega, b \otimes \omega \rangle \\ &= q^{-h_1/2}|_\omega (e_{-1} a, b) = -i(e_{-1} a, b) \end{aligned} \tag{52}$$

because $\langle , \rangle$ is antilinar in the first argument and linear in the second. Therefore

$$\langle a \cdot \omega, e_1(b \cdot \omega) \rangle = -\langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle. \tag{53}$$

Similarly $\langle a \cdot \omega, e_2(b \cdot \omega) \rangle = \langle e_{-2}(a \cdot \omega), b \cdot \omega \rangle$. This shows that x^* is indeed the adjoint of x w.r.t. $\langle \ , \ \rangle$ which is positive definite, because $(\ , \ )$ is positive definite by definition. Theorem (4.5) now completes the proof.

□

As a consistency check, one can see again from section 2 that all the $U_{\tilde{q}}(SO(2,1))$ resp. $U_{\tilde{q}}(SU(2))$ subgroups are unitarizable in these representations, but this is not enough to show the unitarizability for the full group. Note that for $n = 1$, one obtains the classical

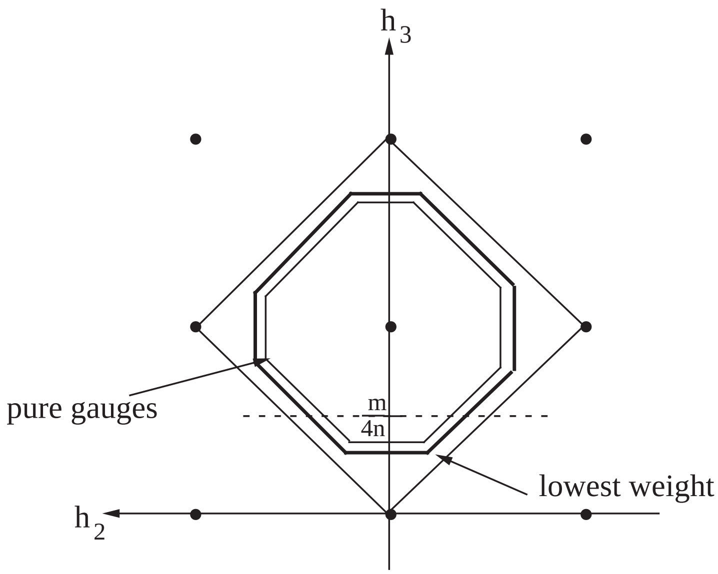

Figure 4: physical representation with subspace of pure gauges (for $\frac{m}{2n}$ integer), schematically. For $h_3 \leq \frac{m}{4n}$, the structure is the same as for $q = 1$.

one–particle representations for given s, E_0 as $m \to \infty$. We have therefore also proved the unitarizability at $q = 1$ for (half)integer spin, which appears to be non trivial in itself [10]. Furthermore, *all representations obtained from the above by shifting E_0 or s by a multiple of $\frac{m}{n}$ are unitarizable as well.* One obtains in weight space a cell–like structure of representations which are unitarizable w.r.t. $U_q(SO(2,3))$ resp. $U_q(SO(5))$.

Finally we want to consider many–particle representations, i.e. find a tensor product such that the tensor product of unitary representations is unitarizable, as in section 2. The idea is the same as there, the tensor product of 2 such representations will be a direct sum of representations, of which we only keep the appropriate physical lowest weight "submodules". To make this more precise, consider two physical irreps $V_{(\mu)}$ and $V_{(\mu')}$ as in Definition 4.6. For a basis $\{u_{\lambda'}\}$ of lowest weight vectors in $V_{(\mu)} \otimes V_{(\mu')}$ with physical λ', consider the linear span $\oplus \mathcal{U} u_{\lambda'}$ of its lowest weight submodules, and let $Q_{\mu,\mu'}$ be the quotient of this after factoring out all proper submodules of the $\mathcal{U} u_{\lambda'}$. Let $\{u_{\lambda''}\}$ be a basis of lowest weight vectors of $Q_{\mu,\mu'}$. Then $Q_{\mu,\mu'} = \oplus V_{(\lambda'')}$ where $V_{(\lambda'')}$ are the corresponding (physical) irreducible lowest weight modules, i.e. $Q_{\mu,\mu'}$ is completely reducible. Now we define the following:

Definition 4.8 *In the above situation, let $\{u_{\lambda''}\}$ be a basis of physical lowest weight states of $Q_{\mu,\mu'}$, and let $V_{(\lambda'')}$ be the corresponding physical lowest weight irreps. Then define*

$$V_{(\mu)} \tilde{\otimes} V_{(\mu')} := \bigoplus_{\lambda''} V_{(\lambda'')} \tag{54}$$

Notice that if $\frac{m}{2n}$ is not an integer, then the physical states have non–integral weights, and the full tensor product of two physical irreps $V_{(\mu)} \otimes V_{(\mu')}$ does not contain any physical lowest weights. Therefore $V_{(\mu)} \tilde{\otimes} V_{(\mu')}$ is zero in that case.

Again as in section 2, one might also include a second "band" of high–energy states.

Theorem 4.9 *If all weights in the factors are integral, then $\tilde{\otimes}$ is associative, and $V_{(\mu)} \tilde{\otimes} V_{(\mu')}$ is unitarizable w.r.t. $U_q(SO(2,3))$.*

Proof First, notice that the λ'' are all integral and none of them gives rise to a massless representation or a singleton. Thus none of the $\mathcal{U}u_{\lambda'}$ contain a physical lowest weight vector according to Proposition 4.2. Also, lowest weight vectors for generic q cannot disappear at roots of unity. Therefore $Q_{\mu,\mu'}$ contains all the physical lowest weight vectors of the full tensor product. Furthermore, no physical lowest weight vectors are contained in products of the form (discarded vectors)$\otimes$ (any vectors). Associativity now follows from the associativity of the full tensor product and the coassociativity of the coproduct, and the stucture for energies $h_3 \leq \frac{m}{4n}$ is the same as classically (observe that since there are no massless representations, classically inequivalent physical representations cannot recombine into indecomposable ones). $\square$

In particular, none of the low–energy states have been discarded. Therefore our definition is physically sensible, and the case of $q = e^{2\pi i/m}$ with m even appears to be most interesting physically.

5 Conclusion

We have shown that in contrast to the classical case, there exist unitary representations of noncompact quantum groups at roots of unity. In particular, we have found finite dimensional unitary representations of $U_q(SO(2,3))$ corresponding to all classical "physical" representations, with the same structure at low energies as in the classical case. Thus they could be used to describe elementary particles with arbitrary spin. This generalizes earlier results of [6] on the singletons. Representations for many non–identical particles are found.

Apart from purely mathematical interest, this is very encouraging for applications in QFT. In particular the appearance of pure gauge states should be a good guideline to construct gauge theories on quantum Anti–de Sitter space. If this is possible, one should expect it to be finite in light of these results. However to achieve that goal, more ingredients are needed, such as implementing a symmetrization axiom (cp. [9]), a dynamical principle (which would presumably involve integration over such a quantum space, cp. [27]), and efficient methods to do calculations in such a context. These are areas of current research.

6 Acknowledgements

It is a pleasure to thank Bruno Zumino for many useful discussions, encouragement and support. I would also like to thank Chong–Sun Chu, Pei–Ming Ho and Bogdan Morariu for useful comments, as well as Paolo Aschieri, Nicolai Reshetikhin, Michael Schlieker and Peter Schupp. In particular, I am grateful to V. K. Dobrev for pointing out the papers [6] to me. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-9514797.

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